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On cohomology of dual Specht modules

V.P. Burichenko,[a] A.S. Kleshchev[b], S. Martin[c,*]

[a] Computer Centre, Kirov St 32a, Gomel 246000, Belarus

[b] Institute of Mathematics, Academy of Sciences of Belarus, Surganova 11, Minsk 220072, Belarus

[c] D.P.M.M.S., 16 Mill Lane, Cambridge, CB2 1SB, UK

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Abstract

Some good cohomological properties of the dual Specht modules for the symmetric group Σ_n are established. These results are used to compute second degree cohomology of some Σ_n-modules over fields of positive characteristic.

0. Introduction

Let Σ_n be the symmetric group on n letters. In this paper we study the cohomology of Σ_n with coefficients in some modules over fields of positive characteristic.

For facts about representations of symmetric groups we refer to James [6]. Irreducible representations of Σ_n over a field of zero characteristic are parametrized by partitions of n (equivalently, by Young diagrams). The irreducible module corresponding to a partition λ is denoted by S^λ and is called the Specht module. In fact, the Specht module can be defined over an arbitrary field. However over a field of positive characteristic, S^λ is not irreducible in general but in a reduction modulo p of the corresponding Specht module in characteristic 0.

Throughout K will denote a field of characteristic $p > 2$ (cf. the remark at the end of the Introduction).

Irreducible $K\Sigma_n$-modules are parametrized by p-regular partitions of n. If λ is p-regular then S^λ has a unique maximal submodule J^λ, and we denote by D^λ the unique irreducible quotient S^λ/J^λ. As λ runs through the set of p-regular partitions of n the modules D^λ form a complete set of inequivalent absolutely irreducible $K\Sigma_n$-modules.

For general facts about group cohomology the reader is referred to Brown [1].

* Corresponding author.

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The main result of this paper is a computation of the groups $H^i(\Sigma_n,(S^\lambda)^*)$ for $i = 1,2$ and p odd.

Theorem 2.4. *$H^1(\Sigma_n,(S^\lambda)^*) = 0$ except the cases $\lambda = (1^3)$, $(n-3,1^3)$ in characteristic three. In the exceptional cases the dimension of H^1 is equal to one.*

For $p > 3$ this theorem is already known from Kleshchev and Martin [9].

Theorem 4.1. *$H^2(\Sigma_n,(S^\lambda)^*) = 0$, except for the following cases in characteristic three: $\lambda = (1^3)$, $(2,1)$, (2^2), (1^6), $(n-3,1^3)$, $(n-3,2,1)$, $(n-6,1^6)$, in which cases $\dim H^2(\Sigma_n,(S^\lambda)^*) = 1$.*

These results allow one to reduce the study of $H^i(\Sigma_n, D^\lambda)$ to that of $H^{i-1}(\Sigma_n,(J^\lambda)^*)$, thus lowering the degree ($i = 1,2$). Using this method we obtain some results on second degree cohomology of Σ_n with coefficients in certain irreducible modules.

Proposition 5.4. *Suppose that $m < p$. Then $H^2(\Sigma_n, D^{(n-m,m)}) = 0$, except the cases:*
(1) *$p = 3$, $n = 3$, $m = 1$ and*
(2) *$p = 3$, $n = 4$, $m = 2$,*
where $\dim H^2(\Sigma_n, D^{(n-m,m)}) = 1$.

Proposition 5.5. $\dim H^2(\Sigma_n, D^{(n-p,p)}) = 0$ *if $n \equiv -1 \pmod p$ and*

$$\dim H^2(\Sigma_n, D^{(n-p,p)}) = 1,$$

otherwise.

We also compute H^2 for irreducible modules associated to hook diagrams, see Proposition 5.3 below. Further results along these lines may be found in the body of the paper.

Unexpectedly good cohomological properties of dual Specht modules may not seem surprising if one takes into account the similarities between the representation theories of symmetric groups and algebraic groups. In the latter theory Weyl modules play a rôle similar to that played by Specht modules in the representation theory of symmetric groups. However, it is well-known that dual Weyl modules are acyclic, cf. [8].

Remark. A number of circumstances conspire to make the case $p = 2$ exceptional. For example both $H^1(\Sigma_n, \mathbb{F}_2)$ and $H^2(\Sigma_n, \mathbb{F}_2)$ are non-zero, giving a counterexample to the main results; further, in characteristic 2 a dual Specht module is again a Specht module.

As for generalizations of the main results to higher degree cohomologies, we prove

Proposition 6.1. *Fix a prime p and a positive integer i. The following two conditions are equivalent*:

(i) $H^1(\Sigma_n,(S^\lambda)^*) = H^2(\Sigma_n,(S^\lambda)^*) = \cdots = H^i(\Sigma_n,(S^\lambda)^*) = 0$ *for any* λ *and* n.

(ii) $H^1(A_n,\mathbb{F}_p) = H^2(A_n,\mathbb{F}_p) = \cdots = H^i(A_n,\mathbb{F}_p) = 0$ *for any* n *(here* A_n *is the alternating group).*

Thus the question reduces to an investigation of higher cohomologies of A_n with trivial coefficients.

1. Preliminaries

Recall that K is a field of characteristic $p > 2$. We need to state some basic facts about representations of Σ_n from James [6, 7].

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ be a (proper) partition of n, i.e. $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0$ and $\sum \lambda_i = n$. We often gather together equal parts and write $\lambda = (\lambda_1^{a_1}, \ldots, \lambda_m^{a_m})$, where $\lambda_1 > \cdots > \lambda_m > 0$, $a_i > 0$ and $\sum a_i\lambda_i = n$. We denote by λ' the partition conjugate to λ, i.e. the diagram of λ' is transposed to that of λ. We write $\lhd$ for the standard partial ('dominance') order on partitions.

Let $\Sigma_\lambda = \Sigma_{\lambda_1} \times \cdots \times \Sigma_{\lambda_k}$ be the standard Young subgroup of Σ_n. Let M^λ be the permutation module $M^\lambda = (1_{\Sigma_\lambda})\uparrow^{\Sigma_n}$. It is known that M^λ has a basis consisting of λ-tabloids (the λ-tableaux with 'unordered row entries', for example as in James [6, 3.13]).

The Specht module S^λ may be defined explicitly as a submodule of M^λ. We shall often write 1 for the trivial module, $S^{(n)}$, and sgn for the sign representation, $S^{(1^n)}$ of Σ_n. If λ is p-regular (so there does not exist an i such that $\lambda_i = \lambda_{i+1} = \ldots = \lambda_{i+p-1}$) then S^λ has a unique irreducible quotient, called D^λ in the Introduction.

The following two facts will often be used without explicit reference.

The modules D^λ are self-dual.

On the other hand, from [6, 8.15], we know that

$$(S^\lambda)^* \cong S^{\lambda'} \otimes \mathrm{sgn}.$$

If λ is p-regular D^λ occurs once as a composition factor of S^λ and all other factors are isomorphic to some D^μ for $\mu \rhd \lambda$; if λ is not p-regular then for any composition factor D^μ of S^λ we have $\mu \rhd \lambda$.

If an Σ_n-module has a filtration

$$(\mathscr{F})\ 0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_N = M$$

with factors $F_i = M_i/M_{i-1}$, we write $M \sim F_1|F_2|\cdots F_N$.

In particular, a filtration $(\mathscr{F})$ is called a Specht filtration if all factors F_i $(1 \leq i \leq N)$ are isomorphic to Specht modules.

Theorem 1.1. *Any M^λ has a Specht filtration with $M_1 \cong S^\lambda$ and for any $i > 1$ one has $F_i \cong S^\mu$ for some $\mu \rhd \lambda$.*

We now recall the Branching Theorem of [6, 9.3]. Given $\lambda = (\lambda_1^{a_1}, \dots, \lambda_m^{a_m})$ with $\lambda_1 > \cdots > \lambda_m$, denote by $\lambda(i)$ the partition

$$(\lambda_1^{a_1}, \dots, \lambda_{i-1}^{a_{i-1}}, \lambda_i^{a_i - 1}, \lambda_i - 1, \lambda_{i+1}^{a_{i+1}}, \dots, \lambda_m^{a_m})$$

of $n - 1$.

Theorem 1.2. *For the restriction $(S^\lambda)\downarrow_{\Sigma_{n-1}}$ one has*

$$(S^\lambda)\downarrow_{\Sigma_{n-1}} \sim S^{\lambda(m)} | S^{\lambda(m-1)} | \dots | S^{\lambda(1)}.$$

Let λ and μ be partitions of m and n, respectively. If the characteristic is zero, the decomposition of the induced module $\mathrm{Ind}_{\Sigma_m \times \Sigma_n}^{\Sigma_{m+n}}(S^\lambda \otimes S^\mu)$ can be deduced from the 'Littlewood–Richardson Rule'. In the case of positive characteristic this module has a Specht filtration with the same factors. Actually we will need only one special case of this (see [6, 17.14]).

Theorem 1.3. *Let λ be a partition of n. Then*

$$\mathrm{Ind}_{\Sigma_m \times \Sigma_n}^{\Sigma_{m+n}}(S^\lambda \otimes 1_{\Sigma_m}) \sim S^{\nu_1} | \dots | S^{\nu_r};$$

here $\nu_1, \dots, \nu_r$ comprise the diagrams ν such that $\nu \supset \lambda$ and all nodes from $\nu \backslash \lambda$ belong to distinct columns. Moreover, if $\nu_i \lhd \nu_j$ then $i < j$.

As a particular case we describe the Specht filtration for

$$\mathrm{Ind}_{\Sigma_{n-j} \times \Sigma_j}^{\Sigma_n}(1_{n-j} \otimes \mathrm{sgn}_j) = I_{n,j} \quad (j = 1, \dots, n-1),$$

where we write 1_{n-j} for $S^{(n-j)}$ and sgn_j for $S^{(1^j)}$. In this case there are precisely two diagrams $[\nu]$, $|[\nu]| = n$ containing $[1^j]$ and such that all nodes from $[\nu]\backslash[1^j]$ belong to distinct columns, namely $(n - j + 1, 1^{j-1})$ and $(n - j, 1^j)$. From Theorem 1.3, we have:

Corollary 1.4. $I_{n,j} \sim S^{(n-j,1^j)} | S^{(n-j+1,1^{j-1})}$ *for* $1 \le j \le n - 1$.

If λ, μ are two partitions of n and $[\lambda]$ is the diagram of λ then the λ-tableau of type μ is the assignment of μ_1 1s, μ_2 2s, etc. in the nodes of $[\lambda]$. A λ-tableau of type μ is called semistandard if the entries in each row are weakly increasing from left to right and the entries in each column are strictly increasing from top to bottom.

Proposition 1.5. *The dimension of the space $\mathrm{Hom}_{\Sigma_n}(S^\lambda, M^\mu)$ is equal to the number of semistandard λ-tableaux of type μ.*

Now recall that a rim p-hook associated to a diagram $[\lambda]$ is a connected part of its edge consisting of p nodes and such that removing it from $[\lambda]$ produces the diagram of (another) partition. The p-core, $\tilde{\lambda}$, of λ is that partition whose diagram is obtained by removing as many rim p-hooks as possible. We use repeatedly the 'Nakayama Conjecture' in the following form.

Theorem 1.6. *S^λ and S^μ lie in the same p-block of Σ_n if and only if $\tilde{\lambda} = \tilde{\mu}$.*

By considering central idempotents it is easy to see that $H^i(\Sigma_n, M) = 0$ $(i > 0)$ for any indecomposable Σ_n-module M not in the principal p-block. Combining this with Theorem 1.6 gives.

Corollary 1.7. *If $\tilde{\lambda} \neq \emptyset$ and $\tilde{\lambda}$ is not a row then $H^i(\Sigma_n, (S^\lambda)^*) = 0$.*

Finally we need to introduce one more set of Σ_n-modules which are indexed by partitions of n. There exists an indecomposable direct summand Y^λ of M^λ containing S^λ, this is called the Young module. There are many references for Young modules, for example [7], [3]. It is known that $Y^\lambda \cong Y^\mu$ if and only if $\lambda = \mu$ and that

Lemma 1.8. *Y^λ is projective if λ' is p-regular.*

Proposition 1.9. *M^λ is isomorphic to a direct sum of Y^λ and other Young modules Y^μ with $\mu \rhd \lambda$.*

Theorem 1.10. *Y^λ has a Specht filtration*

$$0 = Y_0 \subseteq Y_1 \subseteq \cdots \subseteq Y_N = Y^\lambda$$

with $Y_1 \cong S^\lambda$ and $Y_i/Y_{i-1} \cong S^\mu$, $\mu \rhd \lambda$, for $i > 1$.

2. 1-cohomologies of dual Specht modules

In this section we compute the groups $H^1(\Sigma_n, (S^\lambda)^*)$ for $p > 2$. The case $p > 3$ was considered in [9]. Our approach allows us to simplify the proof of the version appearing there and we can also handle the case $p = 3$ completely.

Of course the computation of 0-cohomologies (invariants) is trivial.

Lemma 2.1. *$H^0(\Sigma_n, (S^\lambda)^*) = ((S^\lambda)^*)^{\Sigma_n} = K$ if $\lambda = (n)$ and $H^0(\Sigma_n, (S^\lambda)^*) = 0$* otherwise.

Proof. The case $\lambda = (n)$ is clear since then $S^\lambda \cong K$. Otherwise

$$((S^\lambda)^*)^{\Sigma_n} \cong \mathrm{Hom}_{\Sigma_n}(K, (S^\lambda)^*) \cong \mathrm{Hom}_{\Sigma_n}(S^\lambda, K) = \mathrm{Hom}_{\Sigma_n}(S^\lambda, M^{(n)}).$$

The latter space is zero by James [6, 13.17]. $\square$

We collect together some results on low degree cohomologies of the trivial and sign modules for later use.

Lemma 2.2. (i) $H^1(\Sigma_n, K) = 0$.

(ii) $H^2(\Sigma_n, K) = 0$

(iii) $H^1(\Sigma_n, \mathrm{sgn}) = 0$, *except in the cases* $(n,p) = (3,3)$, $(4,3)$ *in which cases* $\dim H^1(\Sigma_n, \mathrm{sgn}) = 1$.

(iv) $H^2(\Sigma_n, \mathrm{sgn}) = 0$, *except in the cases* $(n,p) = (3,3), (4,3), (6,3), (7,3)$, *in which cases* $\dim H^2(\Sigma_n, \mathrm{sgn}) = 1$.

Proof. For any group $G, H^1(G,K) = \mathrm{Hom}(G,K) = \mathrm{Hom}(G/G', K)$, where G' is the derived subgroup of G. Now $\Sigma_n' = A_n$ and $p > 2$ so (i) follows. Similarly $H^1(A_n, K) = 0$ if $(n,p) \neq (3,3)$ or $(4,3)$, while $H^1(A_n, K) \cong K$ if $(n,p) = (3,3)$ or $(4,3)$. Note the decomposition $(1_{A_n})\uparrow^{\Sigma_n} \cong 1_{\Sigma_n} \oplus \mathrm{sgn}$. Using Shapiro's Lemma [1, III.6.2] we have a decomposition

$$H^1(\Sigma_n, 1_{\Sigma_n}) \oplus H^1(\Sigma_n, \mathrm{sgn}) \cong H^1(\Sigma_n, (1_{A_n})\uparrow^{\Sigma_n}) \cong H^1(A_n, 1_{A_n}).$$

Since $H^1(\Sigma_n, 1_{\Sigma_n}) = 0$ it follows that $H^1(\Sigma_n, \mathrm{sgn}) \cong H^1(A_n, 1_{A_n})$, giving (iii).

Next we consider degree two homology. For (ii) see Huppert [5, 25.12]. Now $H^2(A_n, K) = 0$ except when $(n,p) = (3,3)$, $(4,3)$, $(6,3)$, $(7,3)$ in which cases it is 1-dimensional. Actually for $n \geq 5$ this comes from Gorenstein [4, 4.15], the cases $n = 3, 4$ being straightforward. Now, as in the proof of (iii), one has $H^2(\Sigma_n, \mathrm{sgn}) \cong H^2(A_n, K)$. $\square$

The point now is to use the inclusion $(S^\lambda)^* \hookrightarrow M^{\lambda'} \otimes \mathrm{sgn}$. For we compute cohomologies of the latter module in the following lemma.

Lemma 2.3. *If* $p > 3$ *then* $H^1(\Sigma_n, M^\lambda \otimes \mathrm{sgn}) = H^2(\Sigma_n, M^\lambda \otimes \mathrm{sgn}) = 0$.

If $p = 3$ *then* $H^1(\Sigma_n, M^\lambda \otimes \mathrm{sgn}) = 0$ *except when* $\lambda = (3, 1^{n-3})$, $(4, 1^{n-4})$ *and* $H^2(\Sigma_n, M^\lambda \otimes \mathrm{sgn}) = 0$ *except when* $\lambda = (3, 1^{n-3}), (4, 1^{n-4}), (6, 1^{n-6}), (7, 1^{n-7}), (3^2, 1^{n-6}), (4, 3, 1^{n-7}), (4^2, 1^{n-8})$. *In the exceptional cases the dimensions of* H^i *are all equal to one.*

Proof. By [2, 38.5] one has

$$M^\lambda \otimes \mathrm{sgn} \cong (1_{\Sigma_\lambda})\uparrow^{\Sigma_n} \otimes \mathrm{sgn} \cong (1_{\Sigma_\lambda} \otimes \mathrm{sgn}_{\Sigma_\lambda})\uparrow^{\Sigma_n} \cong (\mathrm{sgn}_{\Sigma_\lambda})\uparrow^{\Sigma_n}.$$

By Shapiro's Lemma $H^i(\Sigma_n, M^\lambda \otimes \mathrm{sgn}) \cong H^i(\Sigma_\lambda, \mathrm{sgn}_{\Sigma_\lambda})$. If $\lambda = (1^n)$ then $\Sigma_\lambda = 1$ and $H^i(\Sigma_\lambda, \mathrm{sgn}_{\Sigma_\lambda}) = 0$. If $\lambda = (n)$ we can quote Lemma 2.2. Thus we may assume that $k > 1$ and $\lambda_1 > 1$.

From the definition of a Young subgroup and the fact that $\mathrm{sgn}_{\Sigma_\lambda} \cong \mathrm{sgn}_{\lambda_1} \otimes \cdots \otimes \mathrm{sgn}_{\lambda_k}$ we have, by the Künneth formula,

$$H^i(\Sigma_\lambda, \mathrm{sgn}_{\Sigma_\lambda}) = \sum H^{i_2}(\Sigma_{\lambda_1}, \mathrm{sgn}_{\lambda_1}) \otimes \cdots \otimes H^{i_k}(\Sigma_{\lambda_k}, \mathrm{sgn}_{\lambda_k}),$$

where the sum runs over all tuples $(i_1, \ldots, i_k) \in \mathbb{Z}_+^k$ such that $i_1 + \cdots + i_k = i$. In particular,

$$H^1(\Sigma_\lambda, \mathrm{sgn}_{\Sigma_\lambda}) = \sum_{j=1}^{k} H^0(\Sigma_{\lambda_1}, \mathrm{sgn}_{\lambda_1}) \otimes \cdots \otimes H^1(\Sigma_{\lambda_j}, \mathrm{sgn}_{\lambda_j}) \otimes \cdots \otimes H^0(\Sigma_{\lambda_k}, \mathrm{sgn}_{\lambda_k}).$$

Observe that the summands of the right-hand side corresponding to $j > 1$ vanish since $H^0(\Sigma_{\lambda_1}, \mathrm{sgn}_{\lambda_1}) = 0$. If $\lambda_2 > 1$ then $H^0(\Sigma_{\lambda_2}, \mathrm{sgn}_{\lambda_2}) = 0$ and all summands vanish. With $\lambda_2 = 1$, $\lambda = (\lambda, 1^{n-\lambda_1})$ and the sum equals $H^1(\Sigma_{\lambda_1}, \mathrm{sgn}_{\lambda_1})$. Now apply Lemma 2.2. The proof for H^2 is analogous. □

We are now in a position to formulate the main result of this section.

Theorem 2.4. *$H^1(\Sigma_n, (S^\lambda)^*) = 0$ except the cases $\lambda = (1^3)$, $(n-3, 1^3)$ in characteristic three. In the exceptional cases the dimension of H^1 is equal to one.*

Proof. Write $Q^\lambda = M^\lambda / S^\lambda$. Tensor the short exact sequence

$$0 \to S^{\lambda'} \to M^{\lambda'} \to Q^{\lambda'} \to 0$$

with the sign module to obtain

$$0 \to S^{\lambda'} \otimes \mathrm{sgn} \to M^{\lambda'} \otimes \mathrm{sgn} \to Q^{\lambda'} \otimes \mathrm{sgn} \to 0. \tag{1}$$

By the long exact sequence in cohomology and [6, 8.15] we have

$$H^0(\Sigma_n, Q^{\lambda'} \otimes \mathrm{sgn}) \to H^1(\Sigma_n, (S^\lambda)^*) \to H^1(\Sigma_n, M^{\lambda'} \otimes \mathrm{sgn}) \to H^1(\Sigma_n, Q^{\lambda'} \otimes \mathrm{sgn}). \tag{2}$$

Since $Q^{\lambda'}$ has a Specht filtration with factors $S^{\mu'}$ where $\mu' \rhd \lambda'$ (cf. 1.1), $Q^{\lambda'} \otimes \mathrm{sgn}$ has a filtration with factors $S^{\mu'} \otimes \mathrm{sgn} \cong (S^\mu)^*$ for $\mu \lhd \lambda$. Now Lemma 2.1 implies that $H^0(\Sigma_n, Q^{\lambda'} \otimes \mathrm{sgn}) = 0$, hence $H^1(\Sigma_n, (S^\lambda)^*)$ embeds into $H^1(\Sigma_n, M^{\lambda'} \otimes \mathrm{sgn})$. Then Lemma 2.3 yields that $H^1(\Sigma_n, (S^\lambda)^*) = 0$, except when $p = 3$ and $\lambda' = (3, 1^{n-3})$ or $(4, 1^{n-4})$, i.e. $\lambda = (n-2, 1^2)$ $(n \geq 3)$ or $\lambda = (n-3, 1^3)$ $(n \geq 4)$.

We now consider the two exceptions separately.

Case: $p = 3$, $\lambda = (n-3, 1^3)$. Now $Q^{\lambda'} \otimes \mathrm{sgn}$ has a filtration with factors $(S^\mu)^*$, $\mu \lhd (n-3, 1^3)$. It follows from the above that for any such factor, $H^1(\Sigma_n, (S^\mu)^*) = 0$ because $\mu \lhd (n-3, 1^3)$ implies that μ is different from both $(n-2, 1^2)$ and $(n-3, 1^3)$. Hence $H^1(\Sigma_n, Q^{\lambda'} \otimes \mathrm{sgn}) = 0$. From the exactness of (2) we have $H^1(\Sigma_n, (S^\lambda)^*) \cong H^1(\Sigma_n, M^{\lambda'} \otimes \mathrm{sgn})$ and an application of Lemma 2.3 completes the proof for the case $\lambda = (n-3, 1^3)$.

Case: $p = 3$, $\lambda = (n-2, 1^2)$. If $n = 3$ apply Lemma 2.2 (iii). Assume that $n \geq 4$. If 3 does not divide n then the 3-core of λ is not a row, so Corollary 1.7 implies that $H^1(\Sigma_n, (S^\lambda)^*) = 0$. Suppose finally that $3 | n$ $(n \geq 6)$. By [9, Section 2] (or from the proof of Theorem 3.2) we have, for any Σ_n-module V, an exact sequence:

$$\mathrm{Hom}_{\Sigma_n}(S^{(n-1,1)}, V) \to H^1(\Sigma_n, V) \to H^1(\Sigma_{n-1}, V\downarrow_{\Sigma_{n-1}}). \tag{3}$$

Taking $V = (S^{(n-2,1^2)})^*$ one obtains $\mathrm{Hom}_{\Sigma_n}(S^{(n-1,1)}, (S^{(n-2,1^2)})^*) = 0$ since $(S^{(n-2,1^2)})^*$ has simple socle isomorphic to $D^{(n-2,1^2)}$, which is not a composition factor of $S^{(n-1,1)}$. By the Branching Theorem of 1.2,

$$(S^{(n-2,1^2)})^*\downarrow_{\Sigma_{n-1}} \sim (S^{(n-2,1)})^* | (S^{(n-3,1^2)})^*.$$

But 3 cannot divide $n-1$ so

$$H^1(\Sigma_{n-1}, (S^{(n-3,1^2)})^*) = H^1(\Sigma_{n-1}, (S^{(n-2,1)})^*) = 0,$$

whence $H^1(\Sigma_{n-1}, (S^{(n-2,1^2)})^*\downarrow_{\Sigma_{n-1}}) = 0$. From the exactness of (3),

$$H^1(\Sigma_n, (S^{(n-2,1^2)})^*) = 0. \qquad \square$$

3. The inductive step

The main result of this section is Theorem 3.2 which allows one to compute $H^2(\Sigma_n, V)$ under the assumption that $H^2(\Sigma_{n-1}, V\downarrow_{\Sigma_{n-1}}) = H^1(\Sigma_{n-2}, V\downarrow_{\Sigma_{n-2}}) = 0$. Taking V to be irreducible gives Proposition 5.7.

We define a certain submodule T in $M^{(n-2,1^2)}$ as follows: $M^{(n-2,1^2)}$ can be thought of as the space spanned by ordered pairs, which for typographical convenience we shall denote by $/i/j/$ $(i \neq j)$ (and with Σ_n-action $(/i/j/)\pi = /i\pi/j\pi/$ as in James [6, p. 7]). Define T as the kernel of the homomorphism

$$\varphi : M^{(n-2,1^2)} \to M^{(n-1,1)}$$
$$/i/j/ \mapsto \bar{\imath} - \bar{\jmath}$$

It is clear that $\mathrm{Im}\,\varphi = S^{(n-1,1)}$.

Lemma 3.1 $T \cong M^{(n-2,2)} \otimes S^{(n-2,1^2)}$.

Proof. Define an involution $\sigma \in \mathrm{End}_{\Sigma_n}(M^{(n-2,1^2)})$ by $\sigma(/i/j/) = /j/i/$. Let $M_\pm$ be the space generated by elements of the form $/i/j/ \pm /j/i/$, $(i \neq j)$, respectively. Clearly $M^{(n-2,1^2)} = M_+ \oplus M_-$ is a decomposition into the eigenspaces of σ. Since $M_+ \subseteq T$, $T = M_+ \oplus (M_- \cap T)$. In fact $M_+ \cong M^{(n-2,2)}$, so it remains to prove that the second summand is isomorphic to $S^{(n-2,1^2)}$.

Firstly, $\varphi S^{(n-2,1^2)} = 0$ since $\mathrm{Hom}_{\Sigma_n}(S^{(n-2,1^2)}, M^{(n-1,1)}) = 0$ by Proposition 1.5, and so $S^{(n-2,1^2)} \subseteq T$. To show that $S^{(n-2,1^2)} \subseteq M_-$, notice that $(1+\sigma)|_{S^{(n-2,1^2)}}$ is a homomorphism from $S^{(n-2,1^2)}$ to $M^{(n-2,2)}$. However, $\mathrm{Hom}_{\Sigma_n}(S^{(n-2,1^2)}, M^{(n-2,2)}) = 0$ hence $\sigma = -1$ on $S^{(n-2,1^2)}$, i.e. $S^{(n-2,1^2)} \subseteq M_-$. Thus $S^{(n-2,1^2)} \subseteq M_- \cap T$.

To complete the proof

$$\begin{aligned}\dim(M_- \cap T) &= \dim T - \dim M_+ \\ &= \dim M^{(n-2,1^2)} - \dim S^{(n-1,1)} - \dim M^{(n-2,2)} \\ &= \frac{(n-1)(n-2)}{2},\end{aligned}$$

which equals $\dim S^{(n-2,1^2)}$ by the 'Hook Formula' [6, Section 20]. □

Theorem 3.2. *Let V be an Σ_n-module and*

$$H^1(\Sigma_{n-2}, V\downarrow_{\Sigma_{n-2}}) = H^2(\Sigma_{n-1}, V\downarrow_{\Sigma_{n-1}}) = 0.$$

Then

$$\begin{aligned}\dim H^2(\Sigma_n, V) &= \dim \mathrm{Hom}_{\Sigma_n}(S^{(n-2,1^2)}, V) + \dim \mathrm{Hom}_{\Sigma_n}(M^{(n-2,2)}, V) \\ &\quad + \dim H^1(\Sigma_n, V) - \dim H^1(\Sigma_{n-1}, V\downarrow_{\Sigma_{n-1}}) - \dim V^{\Sigma_n} \\ &\quad + \dim V^{\Sigma_{n-1}} - \dim V^{\Sigma_{n-2}}.\end{aligned}$$

Proof. Recall the definition of T. The short exact sequences

$$0 \to S^{(n-1,1)} \to M^{(n-1,1)} \to K \to 0,$$

$$0 \to T \to M^{(n-2,1^2)} \to S^{(n-1,1)} \to 0$$

induce long exact sequences

$$\begin{aligned}0 &\to \mathrm{Hom}_{\Sigma_n}(K, V) \to \mathrm{Hom}_{\Sigma_n}(M^{(n-1,1)}, V) \to \mathrm{Hom}_{\Sigma_n}(S^{(n-1,1)}, V) \to \\ &\mathrm{Ext}^1_{\Sigma_n}(K, V) \to \mathrm{Ext}^1_{\Sigma_n}(M^{(n-1,1)}, V) \to \mathrm{Ext}^1_{\Sigma_n}(S^{(n-1,1)}, V) \to \\ &\mathrm{Ext}^2_{\Sigma_n}(K, V) \to \mathrm{Ext}^2_{\Sigma_n}(M^{(n-1,1)}, V) \to\end{aligned} \tag{4}$$

and

$$\begin{aligned}0 &\to \mathrm{Hom}_{\Sigma_n}(S^{(n-1,1)}, V) \to \mathrm{Hom}_{\Sigma_n}(M^{(n-2,1^2)}, V) \to \mathrm{Hom}_{\Sigma_n}(T, V) \to \\ &\mathrm{Ext}^1_{\Sigma_n}(S^{(n-1,1)}, V) \to \mathrm{Ext}^1_{\Sigma_n}(M^{(n-2,1^2)}, V) \to\end{aligned} \tag{5}$$

By Shapiro's Lemma, the definition of the permutation modules and the fact that for $i \geq 0$ $\mathrm{Ext}^i_{\Sigma_n}(K, -) \cong H^i(\Sigma_n, -)$, these sequences may be written as

$$\begin{aligned}0 &\to V^{\Sigma_n} \to V^{\Sigma_{n-1}} \to \mathrm{Hom}_{\Sigma_n}(S^{(n-1,1)}, V) \to H^1(\Sigma_n, V) \to \\ &H^1(\Sigma_{n-1}, V\downarrow_{\Sigma_{n-1}}) \to \mathrm{Ext}^1_{\Sigma_n}(S^{(n-1,1)}, V) \to H^2(\Sigma_n, V) \to 0,\end{aligned}$$

and

$$0 \to \mathrm{Hom}_{\Sigma_n}(S^{(n-1,1)}, V) \to V^{\Sigma_{n-2}} \to \mathrm{Hom}_{\Sigma_n}(T, V) \to \mathrm{Ext}^1_{\Sigma_n}(S^{(n-1)}, V) \to 0.$$

(For example, $\mathrm{Ext}^2_{\Sigma_n}(M^{(n-1,1)}, V) = \mathrm{Ext}^2_{\Sigma_{n-1}}(K, V\downarrow_{\Sigma_{n-1}}) = H^2(\Sigma_{n-1}, V\downarrow_{\Sigma_{n-1}}) = 0$.) Taking dimensions we have that

$$\dim V^{\Sigma_n} - \dim V^{\Sigma_{n-1}} + \dim \mathrm{Hom}_{\Sigma_n}(S^{(n-1,1)}, V) - \dim H^1(\Sigma_n, V)$$
$$+ \dim H^1(\Sigma_{n-1}, V\downarrow_{\Sigma_{n-1}}) - \dim \mathrm{Ext}^1_{\Sigma_n}(S^{(n-1,1)}, V) + \dim H^2(\Sigma_n, V) = 0$$

and

$$\dim \mathrm{Hom}_{\Sigma_n}(S^{(n-1,1)}, V) - \dim V^{\Sigma_{n-2}} + \dim \mathrm{Hom}_{\Sigma_n}(T, V)$$
$$- \dim \mathrm{Ext}^1_{\Sigma_n}(S^{(n-1,1)}, V) = 0.$$

Substituting the expression for $\dim \mathrm{Hom}_{\Sigma_n}(S^{(n-1,1)}, V)$ from the second equation into the first, and recalling Lemma 3.1 gives the result. □

Corollary 3.3. *Let* $H^2(\Sigma_{n-1}, (S^\lambda)^*\downarrow_{\Sigma_{n-1}}) = H^1(\Sigma_{n-2}, (S^\lambda)^*\downarrow_{\Sigma_{n-2}}) = 0$. *Then*

$$\begin{aligned}\dim H^2(\Sigma_n, (S^\lambda)^*) &= \dim H^1(\Sigma_n, (S^\lambda)^*) - \dim H^1(\Sigma_{n-1}, (S^\lambda)^*\downarrow_{\Sigma_{n-1}})\\ &\quad + \dim \mathrm{Hom}_{\Sigma_n}(S^\lambda, (S^{(n-2,1^2)})^*) - \dim \mathrm{Hom}_{\Sigma_n}(S^\lambda, M^{(n)})\\ &\quad + \dim \mathrm{Hom}_{\Sigma_n}(S^\lambda, M^{(n-1,1)}) + \dim \mathrm{Hom}_{\Sigma_n}(S^\lambda, M^{(n-2,2)})\\ &\quad - \dim \mathrm{Hom}_{\Sigma_n}(S^\lambda, M^{(n-2,1^2)}).\end{aligned} \tag{6}$$

Proof. Set $V = (S^\lambda)^*$ in Theorem 3.2. By self-duality of the M^ν, $\mathrm{Hom}_{\Sigma_n}(M^\nu, (S^\lambda)^*) \cong \mathrm{Hom}_{\Sigma_n}(S^\lambda, M^\nu)$ for any partition ν. Also

$$((S^\lambda)^*)^{\Sigma_{n-2}} \cong \mathrm{Hom}_{\Sigma_n}(M^{(n-2,1^2)}, (S^\lambda)^*) \cong \mathrm{Hom}_{\Sigma_n}(S^\lambda, M^{(n-2,1^2)}).$$

A similar argument for the Σ_{n-1} and Σ_n-invariants of $(S^\lambda)^*$ is then applied. □

4. 2-cohomology of dual Specht modules

This section deals with the computation of $H^2(\Sigma_n, (S^\lambda)^*)$ for $p > 2$.

Theorem 4.1. $H^2(\Sigma_n, (S^\lambda)^*) = 0$, *except for the following cases in characteristic three*: $\lambda = (1^3)$, $(2,1)$, (2^2), (1^6), $(n-3, 1^3)$, $(n-3, 2, 1)$, $(n-6, 1^6)$, *in which cases* $\dim H^2(\Sigma_n, (S^\lambda)^*) = 1$.

The proof proceeds via an extended series of lemmas.

Lemma 4.2. *If* $p > 3$ *then* $H^2(\Sigma_n, (S^\lambda)^*) = 0$.

Proof. From sequence (1) we have a long exact sequence

$$H^1(\Sigma_n, Q^{\lambda'} \otimes \mathrm{sgn}) \to H^2(\Sigma_n, (S^{\lambda})^*) \to H^2(\Sigma_n, M^{\lambda'} \otimes \mathrm{sgn}) \to H^2(\Sigma_n, Q^{\lambda'} \otimes \mathrm{sgn}). \tag{7}$$

By Lemma 2.3, $H^2(\Sigma_n, M^{\lambda'} \otimes \mathrm{sgn}) = 0$ for $p > 3$. Furthermore, we know that $Q^{\lambda'} \otimes \mathrm{sgn}$ has a filtration with factors of the form $(S^{\mu})^*$, hence Theorem 2.4 implies that $H^1(\Sigma_n, Q^{\lambda'} \otimes \mathrm{sgn}) = 0$. An appeal to the exactness of sequence (7) completes the proof. $\square$

For the rest of this section we will assume that $p = 3$.

Lemma 4.3. *$H^2(\Sigma_n, (S^{\lambda})^*) = 0$, except perhaps for the following cases: $\lambda = (n)$, $(n-1,1), (n-2,2), (n-3,3), (n-3,2,1), (n-2,1^2), (n-3,1^3), (n-5,1^5), (n-6,1^6)$, $(n-4,2^2)$, $(n-5,2^2,1)$, $(n-6,2^3)$.*

Proof. From the exactness of (7) we deduce that if $H^2(\Sigma_n, (S^{\lambda})^*) \neq 0$ then either $H^2(\Sigma_n, M^{\lambda'} \otimes \mathrm{sgn}) \neq 0$ or $H^1(\Sigma_n, Q^{\lambda'} \otimes \mathrm{sgn}) \neq 0$. The former is possible only in the case where λ' is one of the partitions listed in Lemma 2.3, and this leads to $\lambda \in \{(n-2,1^2), (n-3,1^3), (n-5,1^5), (n-6,1^6), (n-4,2^2), (n-5,2^2,1), (n-6,2^3)\}$. Again, $Q^{\lambda'} \otimes \mathrm{sgn}$ has a dual Specht filtration with factors of the form $(S^{\mu})^*$ for $\mu \lhd \lambda$. Therefore by Theorem 2.4, if $\lambda \not\trianglerighteq (n-3,1^3)$ and $\lambda \not\trianglerighteq (1^3)$ for $n = 3$ then $H^1(\Sigma_n, Q^{\lambda'} \otimes \mathrm{sgn}) = 0$, yielding the first six exceptions. $\square$

Now consider the exceptional cases in Lemma 4.3, the first being trivial by virtue of Lemma 2.2(ii).

Lemma 4.4. *$H^2(\Sigma_n, (S^{(n-1,1)})^*) = 0$, except when $\lambda = (2,1)$.*

Proof. For $n \geq 5$ this has been proved in [10, Theorem 1] (including the case $p = 2$, which is less trivial). For the sake of completeness and convenience, we give a full proof.

Take $n = 3$. Clearly $(S^{(2,1)})^* \sim \mathrm{sgn}|K$. There is an exact sequence

$$H^1(\Sigma_3, K) \to H^2(\Sigma_3, \mathrm{sgn}) \to H^2(\Sigma_3, (S^{(2,1)})^*) \to H^2(\Sigma_3, K),$$

the first and last parts being zero by Lemma 2.2. Another appeal to Lemma 2.2 completes the proof in this case.

If $3 \nmid n$ then $S^{(n-1,1)}$ is not in the principal block, so by Corollary 1.7, $H^2(\Sigma_n, (S^{(n-1,1)})^*) = 0$.

Finally, take $3|n, n \geq 6$. By Theorem 1.2, $(S^{(n-1,1)})^* \downarrow_{\Sigma_{n-1}}$ and $(S^{(n-1,1)})^* \downarrow_{\Sigma_{n-2}}$ have filtrations with factors of the form $(S^{(r)})^*$ and $(S^{(r-1,1)})^*$ for $r = n-1, n-2$, respectively. Since 3 does not divide $n-1$ or $n-2$,

$$H^2(\Sigma_{n-1},(S^{(n-1,1)})^*\downarrow_{\Sigma_{n-1}}) = H^1(\Sigma_{n-2},(S^{(n-1,1)})^*\downarrow_{\Sigma_{n-2}}) = 0.$$

Thus we are in the situation of Corollary 3.3; then by Theorem 2.4

$$H^1(\Sigma_n,(S^{(n-1,1)})^*) = H^1(\Sigma_{n-1},(S^{(n-1,1)})^*\downarrow_{\Sigma_{n-1}}) = 0.$$

To prove that $\mathrm{Hom}_{\Sigma_n}(S^{(n-1,1)},(S^{(n-2,1^2)})^*) = 0$ it suffices to note that the socle of $(S^{(n-2,1^2)})^*$, i.e. $D^{(n-2,1^2)}$ is not a composition factor of $S^{(n-1,1)}$. The last four terms on the right-hand side of expression (6) are equal to $0,1,1,2$ respectively, via Proposition 1.5. For example $\dim \mathrm{Hom}_{\Sigma_n}(S^{(n-1,1)},M^{(n-2,1^2)}) = 2$, as there are exactly two semistandard $(n-1,1)$-tableaux of type $(n-2,1^2)$. □

Lemma 4.5. $H^2(\Sigma_n,(S^{(n-2,2)})^*) = 0$, *except when* $\lambda = (2^2)$.

Proof. Let $\lambda = (2^2)$. Now Σ_4 has a normal subgroup N isomorphic to the Klein Four group which acts trivially on $S^{(2^2)}$, and the corresponding representation of the quotient $\Sigma_4/N \cong \Sigma_3$ is $S^{(2,1)}$. Thus

$$H^2(\Sigma_4,(S^{(2^2)})^*) \cong H^2(\Sigma_3,(S^{(2,1)})^*) \cong K$$

by Lemma 4.4.

Let $n > 4$. If $n \not\equiv 1 \pmod 3$ then $H^2(\Sigma_n,(S^{(n-2,2)})^*) = 0$ by Corollary 1.7. If $n \equiv 1 \pmod 3$ ($n \geq 7$) then 1.2 gives a filtration $(S^{(n-2,2)})^*\downarrow_{\Sigma_{n-1}} \sim (S^{(n-3,2)})^* | (S^{(n-2,1)})^*$. By Lemma 4.4 $H^2(\Sigma_{n-1},(S^{(n-2,1)})^*) = 0$. Moreover, since $n-1 \not\equiv 1 \pmod 3$,

$$H^2(\Sigma_{n-1},(S^{(n-3,2)})^*) = 0.$$

Therefore, $H^2(\Sigma_{n-1},(S^{(n-2,2)})^*\downarrow_{\Sigma_{n-1}}) = 0$. Similarly we have that

$$H^1(\Sigma_{n-2},(S^{(n-2,2)})^*\downarrow_{\Sigma_{n-2}}) = 0.$$

Now apply Corollary 3.3. From Theorem 2.4 and Theorem 1.2 one has

$$H^1(\Sigma_n,(S^{(n-2,2)})^*) = H^1(\Sigma_{n-1},(S^{\lambda})^*\downarrow_{\Sigma_{n-1}}) = 0.$$

Furthermore, as in Lemma 4.4, $\mathrm{Hom}_{\Sigma_n}(S^{(n-2,2)},(S^{(n-2,1^2)})^*) = 0$. The last four members of expression (6) give $-0+0+1-1=0$ (by Proposition 1.5). □

Lemma 4.6. $H^2(\Sigma_n,(S^{(n-3,3)})^*) = 0$.

Proof. Apply induction on n using Corollary 3.3. Let $n = 6$. Then $(S^{\lambda})^*\downarrow_{\Sigma_5} \cong (S^{(3,2)})^*$ so Lemma 4.5 implies that $H^2(\Sigma_5,(S^{(3,2)})^*) = 0$. Moreover $H^1(\Sigma_4,(S^{(3^2)})^*\downarrow_{\Sigma_4}) = 0$ by Theorems 1.2 and 2.4. Similarly as in Lemmas 4.4 and 4.5, one can check that all the members of the right-hand side of (6) are zero. Hence $H^2(\Sigma_n,(S^{(3^2)})^*) = 0$.

For $n > 6$, that $H^2(\Sigma_{n-1}, (S^{(n-3,3)})^* \downarrow_{\Sigma_{n-1}}) = 0$ follows from the inductive hypothesis together with Lemma 4.5. The rest is as for $n = 6$. □

Lemma 4.7. $H^2(\Sigma_n, (S^{(n-2,1^2)}*)) = 0$ *for* $n \geq 4$.

Proof. If $3 \nmid n$ then the cohomologies of $(S^{(n-2,1^2)})^*$ vanish by Corollary 1.7. If $3 | n$, then $n \geq 6$. By the block-theoretic argument used several times before (and since $H^1(\Sigma_{n-2}, 1_{\Sigma_{n-2}}) = 0$), we have the assumptions of Corollary 3.3. Hence $\dim H^2(\Sigma_n, (S^{(n-2,1^2)})^*) = 0 - 0 + 1 - 0 + 0 + 0 - 1 = 0$. For example, that $\dim \operatorname{Hom}_{\Sigma_n}(S^{(n-2,1^2)}, (S^{(n-2,1^2)})^*) = 1$ follows from the fact that $D^{(n-2,1^2)}$ occurs once as a factor of both $S^{(n-2,1^2)}$ and its dual (as the head of the former and the socle of the latter). □

Lemma 4.8. (i) $\dim H^2(\Sigma_n, (S^\lambda)^*) = 1$ *for* $\lambda = (n-3, 1^3), (n-6, 1^6)$.
(ii) $H^2(\Sigma_n, (S^\lambda)^*) = 0$ *for* $\lambda = (n-5, 1^5)$, $3 \nmid n$.

Proof. In view of Lemma 2.2(iv) we may assume that $n \geq 5$. We set

$$I_j = \operatorname{Ind}_{\Sigma_{n-j} \times \Sigma_j}^{\Sigma_n}(1_{n-j} \otimes \operatorname{sgn}_j)$$

for $1 \leq j \leq n-1$. According to Corollary 1.4

$$I_j \sim S^{(n-j,1^j)} | S^{(n-j+1,1^{j-1})}.$$

Clearly I_j is self-dual (cf. [2, 43.9]). Thus,

$$I_j \sim (S^{(n-j+1,1^{j-1})})^* | (S^{(n-j,1^j)})^*.$$

We want to compute $H^2(\Sigma_n, I_j)$.

By Shapiro's Lemma, the Künneth theorem, and Lemma 2.2(iv),

$$\begin{aligned} H^2(\Sigma_n, I_j) &\cong H^2(\Sigma_{n-j} \times \Sigma_j, 1_{n-j} \otimes \operatorname{sgn}_j) \\ &\cong H^2(\Sigma_{n-j}, 1_{n-j}) \otimes H^0(\Sigma_j, \operatorname{sgn}_j) \oplus H^1(\Sigma_{n-j}, 1_{n-j}) \otimes H^1(\Sigma_j, \operatorname{sgn}_j) \\ &\quad \oplus H^0(\Sigma_{n-j}, 1_{n-j}) \otimes H^2(\Sigma_j, \operatorname{sgn}_j) \\ &\cong H^2(\Sigma_j, \operatorname{sgn}_j) \cong \begin{cases} K & \text{if } j = 3, 4, 6, 7, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Consider now the exact sequence

$$\begin{aligned} H^1(\Sigma_n, (S^{(n-j,1^j)})^*) &\to H^2(\Sigma_n, (S^{(n-j+1,1^{j-1})})^*) \to H^2(\Sigma_n, I_j) \\ &\to H^2(\Sigma_n, (S^{(n-j,1^j)})^*). \end{aligned} \tag{8}$$

If $j = 4$ one has, from Theorem 2.4, Lemma 4.3 and (8) that $H^2(\Sigma_n, (S^{(n-3,1^3)})^*) \cong H^2(\Sigma_n, I_4) \cong K$. Similarly $H^2(\Sigma_n, (S^{(n-6,1^6)})^*) \cong K$ for $n > 7$, while if $n = 7$ the last equality follows from Lemma 2.2(iv). This proves (i). Part (ii) follows from Corollary 1.7.

Lemma 4.9. $H^2(\Sigma_n, (S^{(n-4,2^2)})^*) = 0$.

Proof. Let $\lambda = (n-4, 2^2)$. We consider the embedding $S^{\lambda'} \hookrightarrow Y^{\lambda'}$ (cf. 1.10). Let $\tilde{Q}^{\mu} = Y^{\mu}/S^{\mu}$. Since $S^{\lambda'} \otimes \mathrm{sgn} \cong (S^{\lambda})^*$, one has the exact sequence

$$0 \to (S^{\lambda})^* \to Y^{\lambda'} \otimes \mathrm{sgn} \to \tilde{Q}^{\lambda'} \otimes \mathrm{sgn} \to 0,$$

which yields the exact cohomological sequence

$$H^1(\Sigma_n, \tilde{Q}^{\lambda'} \otimes \mathrm{sgn}) \to H^2(\Sigma_n, (S^{\lambda})^*) \to H^2(\Sigma_n, Y^{\lambda'} \otimes \mathrm{sgn}).$$

From Theorem 1.10, $\tilde{Q}^{\mu}$ has a Specht filtration with factors of the form S^{ν}, $\nu \rhd \mu$. So $\tilde{Q}^{\lambda'} \otimes \mathrm{sgn}$ has a filtration with factors of the form $(S^{\nu})^*$ with $\nu \lhd \lambda$. So Theorem 2.4 forces $H^1(\Sigma_n, \tilde{Q}^{\lambda'} \otimes \mathrm{sgn}) = 0$.

We have to distinguish two cases: $n > 6$ and $n = 6$.

For $n > 6$, since $(\lambda')' = \lambda$ is 3-regular, using Lemma 1.8, $Y^{\lambda'}$ is projective, hence so too is $Y^{\lambda'} \otimes \mathrm{sgn}$. Thus $H^2(\Sigma_n, Y^{\lambda'} \otimes \mathrm{sgn}) = 0$. Hence $H^2(\Sigma_n, (S^{\lambda})^*) = 0$ for $n > 6$.

Finally we consider $n = 6$. We will show that $H^2(\Sigma_n, Y^{\lambda'} \otimes \mathrm{sgn}) = 0$. We prove now that $M^{\lambda'} = M^{(3^2)}$ has a trivial direct summand. Let I be the 1-dimensional submodule of $M^{\lambda'}$ generated by ℓ, where ℓ is the sum of all distinct tabloids. Denote by ℓ' the map from $M^{(3^2)}$ to K sending any tabloid to 1. Clearly, $\ell'(\ell) = \binom{6}{3} = 20 \not\equiv 0 \pmod 3$ hence $M^{(3^2)} \cong I \oplus \mathrm{Ker}(\ell')$. Since $Y^{(3^2)}$ and I are indecomposable direct summands of $M^{(3^2)}$ one can write

$$M^{(3^2)} = Y^{(3^2)} \oplus I \oplus X$$

for some submodule X. From Lemma 2.3, $H^2(\Sigma_6, M^{(3^2)} \otimes \mathrm{sgn}) \cong K$ and $H^2(\Sigma_6, I \otimes \mathrm{sgn}) \simeq K$ by Lemma 2.2(iv). Hence $H^2(\Sigma_6, Y^{(3^2)} \otimes \mathrm{sgn}) = 0$. $\square$

Lemma 4.10. $H^2(\Sigma_n, (S^{(n-5,2^2,1)})^*) = 0$.

Proof. The proof is analogous to Lemma 4.9. For $n > 7$ we use the 3-regularity of λ and for $n = 7$ one observes that $3 \nmid \binom{7}{3}$. $\square$

Lemma 4.11. $H^2(\Sigma_n, (S^{(n-6,2^3)})^*) = 0$.

Proof. The method is to reduce to the case $n = 8$. So suppose that $n > 8$ and consider the module $I_n = \mathrm{Ind}_{\Sigma_8 \times \Sigma_{n-8}}^{\Sigma_n}(S^{(2^4)} \otimes 1_{n-8})$. By Theorem 1.3, and taking duals we have filtrations:

$$I_n \sim (S^{(n-8,2^4)} \mid S^{(n-7,2^3,1)} \mid S^{(n-6,2^3)}$$

$$I_n^* \sim (S^{(n-6,2^3)})^* \mid (S^{(n-7,2^3,1)})^* \mid (S^{(n-8,2^4)})^*.$$

Set $X = I_n^*/S^{(n-6,2^3)}$. We show that the first and third members of the exact sequence

$$H^1(\Sigma_n, X) \to H^2(\Sigma_n, (S^{(n-6,2^3)})^*) \to H^2(\Sigma_n, I_n^*)$$

vanish. The first one is trivial using Theorem 2.4. Further we use Shapiro's lemma and the Künneth formula to derive isomorphisms

$$\begin{aligned} H^2(\Sigma_n, I_n^*) &\cong H^2(\Sigma_8 \times \Sigma_{n-8}, (S^{(2^4)})^* \otimes 1_{n-8}) \\ &\cong H^2(\Sigma_8, (S^{(2^4)})^*) \otimes H^0(\Sigma_{n-8}, 1_{n-8}) \\ &\oplus H^1(\Sigma_8, (S^{(2^4)})^*) \otimes H^1(\Sigma_{n-8}, 1_{n-8}) \oplus H^0(\Sigma_8, (S^{(2^4)})^*) \\ &\otimes H^2(\Sigma_{n-8}, 1_{n-8}) \cong H^2(\Sigma_8, (S^{(2^4)})^*). \end{aligned}$$

Let us compute the last cohomology group. We will prove first that $M^{(7,1)}$ is a direct summand of $M^{(4^2)}$ and then follow the procedure of Lemma 4.9.

Consider the homomorphisms $\varphi : M^{(4^2)} \to M^{(7,1)}$ and $\psi : M^{(7,1)} \to M^{(4^2)}$ defined by $\varphi(\overline{abcd}) = \bar{a} + \bar{b} + \bar{c} + \bar{d}$ and $\psi(\bar{a}) = \Sigma\, \overline{abcd}$, where the latter sum runs over all three-element subsets $\{b, c, d\} \subseteq \{1, 2, \ldots, 8\} \setminus \{a\}$ (we have written an $(n-m, m)$-tabloid in the form $\overline{i_1 \cdots i_m}$ for convenience). Now

$$\begin{aligned} \varphi\psi(\bar{1}) &= \varphi\left(\sum_{1<i<j<k} \overline{1ijk}\right) = \sum_{1<i<j<k} (\bar{1} + \bar{\imath} + \bar{\jmath} + \bar{k}) \\ &= \binom{7}{3}\bar{1} + \binom{6}{2}(\bar{2} + \bar{3} + \cdots + \bar{8}) = -\bar{1}. \end{aligned}$$

Hence $\varphi\psi = -\mathrm{id}_{M^{(7,1)}}$. Therefore,

$$M^{(4^2)} = \operatorname{Im}\psi \oplus \operatorname{Ker}\varphi \cong M^{(7,1)} \oplus \operatorname{Ker}\varphi.$$

Since $M^{(7,1)} \cong S^{(7,1)} \oplus K$ we have a decomposition

$$M^{(4^2)} \cong Y^{(4^2)} \oplus S^{(7,1)} \oplus X$$

for some submodule X. Using Lemma 2.3 we have

$$\begin{aligned} K &\cong H^2(\Sigma_8, M^{(4^2)} \otimes \mathrm{sgn}) \cong H^2(\Sigma_8, Y^{(4^2)} \otimes \mathrm{sgn}) \\ &\oplus H^2(\Sigma_8, S^{(7,1)} \otimes \mathrm{sgn}) \oplus H^2(\Sigma_8, X \otimes \mathrm{sgn}). \end{aligned}$$

Since $H^2(\Sigma_8, S^{(7,1)} \otimes \mathrm{sgn}) \cong K$ by Lemma 4.8(i), we obtain $H^2(\Sigma_8, Y^{(4^2)} \otimes \mathrm{sgn}) = 0$, whence $H^2(\Sigma_8, (S^{(2^4)})^*) = 0$ (using a similar method to Lemma 4.9). We are done. □

Lemma 4.12. *If* $3 \mid n$ *and* $n > 6$ *then* $H^2(\Sigma_n, (S^{(n-5,1^5)})^*) = 0$.

Proof. We will prove that $M^{(7,1^{n-7})}$ is a direct summand of $M^{(6,1^{n-6})}$. Consider the homomorphisms $\varphi : M^{(7,1^{n-7})} \to M^{(6,1^{n-6})}$ and $\psi : M^{(6,1^{n-6})} \to M^{(7,1^{n-7})}$ defined by

$$\varphi : \begin{array}{c} \overline{\underline{i_1 \dots i_7}} \\ i_8 \\ \vdots \\ \overline{\underline{i_n}} \end{array} \mapsto \sum_{j=1}^{7} \begin{array}{l} \overline{\underline{i_1 \dots \hat{i_j} \dots i_7}} \\ i_8 \\ \vdots \\ \overline{\underline{i_n}} \\ \overline{\underline{i_j}} \end{array},$$

$$\psi : \begin{array}{c} \overline{\underline{i_1 \dots i_6}} \\ i_7 \\ \vdots \\ \overline{\underline{i_n}} \end{array} \mapsto \begin{array}{l} \overline{\underline{i_1 \dots \dots i_6 i_n}} \\ i_7 \\ \vdots \\ \overline{\underline{i_{n-1}}} \end{array},$$

One can now check that $\psi\varphi = 7.\mathrm{id}_{M^{(7,1^{n-7})}} = \mathrm{id}_{M^{(7,1^{n-7})}}$. Hence $M^{(7,1^{n-7})}$ is a direct summand of $M^{(6,1^{n-6})}$. Moreover, $Y^{(6,1^{n-6})}$ cannot be a direct summand of $M^{(7,1^{n-7})}$ by virtue of Proposition 1.9. Hence we can write

$$M^{(6,1^{n-6})} \cong Y^{(6,1^{n-6})} \oplus M^{(7,1^{n-7})} \oplus X.$$

The argument is completed as in Lemma 4.11. □

For the last part we need the following intermediate step.

Lemma 4.13. *The module $M^{(3,2,1^{n-5})}$ has a Specht filtration*

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_N = M^{(3,2,1^{n-5})}$$

satisfying the following conditions:

(i) $M_1 \cong S^{(3,2,1^{n-5})}$;

(ii) *There exists exactly one r with $M_r/M_{r-1} \cong S^{(4,1^{n-4})}$*;

(iii) *For any $i<r$, the subquotient $M_i/M_{i-1} \cong S^{\mu}$ with $\mu_1 \geq 3$, $\mu_2 \geq 2$, $\mu \rhd (3,2,1^{n-5})$*;

(iv) *For any $i>r$ one has $M_i/M_{i-1} \cong S^{\mu}$ with $\mu \rhd (4,1^{n-4})$.*

Proof. We use results and notation from [6, Section 17]. Begin with the filtration

$$0 \subseteq S^{(3,2),(3,2,1^{n-5})} \subseteq S^{(3,1),(3,2,1^{n-5})} \subseteq S^{(3),(3,2,1^{n-5})} = M^{(3,2,1^{n-5})}.$$

Using James [6, 17.13] the quotients are

$$S^{(3,1),(3,2,1^{n-5})}/S^{(3,2),(3,2,1^{n-5})} \cong S^{(4,1),(4,1^{n-4})};$$

$$S^{(3),(3,2,1^{n-5})}/S^{(3,1),(3,2,1^{n-5})} \cong S^{(5),(5,1^{n-5})} \cong M^{(5,1^{n-5})}.$$

By the same result of James we conclude that $S^{(3,2),(3,2,1^{n-5})}$ has a Specht filtration $0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_{r-1} = S^{(3,2),(3,2,1^{n-5})}$ such that $M_1 \cong S^{(3,2,1^{n-5})}$ and all other quotients have the form S^μ with $\mu_1 \geq 3$, $\mu_2 \geq 2$, $\mu \rhd (3,2,1^{n-5})$.

Similarly $S^{(4,1),(4,1^{n-4})}$ has a Specht filtration $0 = N_0 \subseteq N_1 \subseteq \cdots \subseteq N_s = S^{(4,1),(4,1^{n-4})}$ such that $N_1 \cong S^{(4,1^{n-4})}$ and all other quotients have the form S^μ with $\mu \rhd (4,1^{n-4})$. Finally, $M^{(5,1^{n-5})}$ has a Specht filtration with factors S^μ such that $\mu \rhd (5,1^{n-5})$. The union of these three filtrations gives the filtration desired. $\square$

Lemma 4.14. $H^2(\Sigma_n, (S^{(n-3,2,1)})^*) \cong K$.

Proof. The short exact sequence

$$0 \to (S^{(n-3,2,1)})^* \to M^{(3,2,1^{n-5})} \otimes \mathrm{sgn} \to Q^{(3,2,1^{n-5})} \otimes \mathrm{sgn} \to 0$$

induces an exact sequence

$$\begin{aligned} &H^1(\Sigma_n, M^{(3,2,1^{n-5})} \otimes \mathrm{sgn}) \to H^1(\Sigma_n, Q^{(3,2,1^{n-5})} \otimes \mathrm{sgn}) \\ &\quad \to H^2(\Sigma_n, (S^{(n-3,2,1)})^*) \to H^2(\Sigma_n, M^{(3,2,1^{n-5})} \otimes \mathrm{sgn}), \end{aligned}$$

the first and last members of which are trivial by Lemma 2.3. So $H^2(\Sigma_n, (S^{(n-3,2,1)})^*) \cong H^1(\Sigma_n, Q^{(3,2,1^{n-5})} \otimes \mathrm{sgn})$. We will prove that the last cohomology group has dimension one

From Lemma 4.13 we conclude that there is a filtration

$$0 = J_1 \subseteq J_2 \subseteq \cdots \subseteq J_r \subseteq \cdots \subseteq J_N = Q^{(3,2,1^{n-5})} \otimes \mathrm{sgn}$$

such that the following conditions hold:

(i) $J_i/J_{i-1} \cong (S^{(n-3,1^3)})^*$ if and only if $i = r$;

(ii) For $i < r$ one has $J_i/J_{i-1} \cong (S^\mu)^*$ with $\mu \lhd (n-3,2,1)$, $\mu_1 \geq 2$, $\mu_2 \geq 2$, $\mu_3 \geq 1$;

(iii) For $i > r$ one has $J_i/J_{i-1} \cong (S^\mu)^*$ with $\mu \lhd (n-3,1^3)$.

Begin by computing $H^1(\Sigma_n, J_r)$. By Theorem 2.4 and (ii), $H^1(\Sigma_n, J_{r-1}) = 0$. Moreover, it follows from the cases already considered in this section that $H^2(\Sigma_n, J_{r-1}) = 0$. Considering the exact sequence

$$H^1(\Sigma_n, J_{r-1}) \to H^1(\Sigma_n, J_r) \to H^1(\Sigma_n, J_r/J_{r-1}) \to H^2(\Sigma_n, J_{r-1})$$

we deduce that $H^1(\Sigma_n, J_r) \cong H^1(\Sigma_n, J_r/J_{r-1}) \cong K$ by Theorem 2.4, since $J_r/J_{r-1} \cong (S^{(n-3,1^3)})^*$.

Finally, we have an exact sequence

$$H^0(\Sigma_n J_N/J_r) \to H^1(\Sigma_n, J_r) \to H^1(\Sigma_n, J_N) \to H^1(\Sigma_n J_N/J_r).$$

By property (iii) above and Lemma 2.1 we know that $H^0(\Sigma_n, J_N/J_r) = 0$. By (iii) again and Theorem 2.4 we have $H^1(\Sigma_n, J_N/J_r) = 0$ too. Hence

$$H^1(\Sigma_n, J_N) \cong H^1(\Sigma_n, J_r) \cong K. \qquad \square$$

5. Some applications

In this section we show how the results of Sections 2–4 can be applied to the computation of cohomologies of symmetric groups with coefficients in simple modules.

Proposition 5.1. *Assume that D^μ belongs to the head of S^λ.*

(i) *If $H^1(\Sigma_n,(S^\lambda)^*)=0$ then $H^1(\Sigma_n,D^\mu)\cong H^0(\Sigma_n,(S^\lambda)^*/D^\mu)$. In particular, if λ is p-regular $H^1(\Sigma_n,D^\lambda)\cong H^0(\Sigma_n,(J^\lambda)^*)$.*

(ii) *If $H^1(\Sigma_n,(S^\lambda)^*)=H^2(\Sigma_n,(S^\lambda)^*)=0$ then $H^2(\Sigma_n,D^\mu)\cong H^1(\Sigma_n,(S^\lambda)^*/D^\mu)$. In particular, if λ is p-regular and $\lambda\neq(2,1)$, (2^2), $(n-3,2,1)$ in characteristic three, then $H^2(\Sigma_n,D^\lambda)\cong H^1(\Sigma_n,(J^\lambda)^*)$.*

Proof. If $\lambda=(n)$ then $D^\mu\cong S^\lambda\cong K$ and we just use Lemma 2.2. Let $\lambda\neq(n)$. Since D^μ is a quotient of S^λ it is a submodule of $(S^\lambda)^*$. Thus we have an exact sequence

$$0\to H^0(\Sigma_n,D^\mu)\to H^0(\Sigma_n,(S^\lambda)^*)\to H^0(\Sigma_n,(S^\lambda)^*/D^\mu)$$

$$\to H^1(\Sigma_n,D^\mu)\to H^1(\Sigma_n,(S^\lambda)^*)\to H^1(\Sigma_n,(S^\lambda)^*/D^\mu)$$

$$\to H^2(\Sigma_n,D^\mu)\to H^2(\Sigma_n,(S^\lambda)^*).$$

From Lemma 2.1, $H^0(\Sigma_n,(S^\lambda)^*)=0$, so since $H^1(\Sigma_n,(S^\lambda)^*)=0$ we have an isomorphism $H^1(\Sigma_n,D^\mu)\cong H^0(\Sigma_n,(S^\lambda)^*/D^\mu)$. To get (i) just notice that, by Theorem 2.4, $H^1(\Sigma_n,(S^\lambda)^*)=0$ provided λ is p-regular.

Now the equalities $H^1(\Sigma_n,(S^\lambda)^*)=H^2(\Sigma_n,(S^\lambda)^*)=0$ would give us an isomorphism

$$H^2(\Sigma_n,D^\mu)\cong H^1(\Sigma_n,(S^\lambda)^*/D^\mu).$$

But if λ is p-regular and $\lambda\neq(2,1)$, (2^2), $(n-3,2,1)$ in characteristic three, then, Theorems 2.4 and 4.1 give us $H^1(\Sigma_n,(S^\lambda)^*)=H^2(\Sigma_n,(S^\lambda)^*)=0$. □

We now consider hook partitions. By Peel [11], if $p\nmid n$, the Specht modules $S^{(n-j,1^j)}$ $(0\leq j\leq n-1)$ are pairwise distinct irreducible modules. We set $h_{n,j}=S^{(n-j,1^j)}$ for $p\nmid n$. If $p|n$ then Peel's result says that there exist pairwise distinct simple modules $h_{n,k}$ $(0\leq k\leq n-2)$ such that the Specht modules $S^{(n-j,1^j)}$ $(0\leq j\leq n-1)$ has a submodule $h_{n,j-1}$ and $S^{(n-j,1^j)}/h_{n,j-1}\cong h_{n,j}$ (interpreting $h_{n,-1}$ and $h_{n,n-1}$ as zero). Of course, if $(n-j,1^j)$ is p-regular then $h_{n,j}\cong D^{(n-j,1^j)}$.

Proposition 5.2. $H^1(\Sigma_n,h_{n,j})=0$, *except when*

(1) $p|n$, $j=1$;

(2) $p=3$, $j=3$.

In the exceptional cases $\dim H^1(\Sigma_n,h_{n,j})=1$.

Proof. If $p \nmid n$ then, since $h_{n,j} \cong S^{(n-j,1^j)} \cong (S^{(n-j,1^j)})^*$, the result comes from Theorem 2.4.

Let $p|n$, $p > 3$. By Theorem 2.4, $H^1(\Sigma_n, (S^{(n-j,1^j)})^*) = 0$. By Proposition 5.1

$$H^1(\Sigma_n, h_{n,j}) \cong H^0(\Sigma_n, h_{n,j-1}) = \begin{cases} 0, & j \neq 1, \\ K, & j = 1. \end{cases}$$

Here we have used the isomorphism $(S^{(n-j,1^j)})^*/h_{n,j} \cong h_{n,j-1}$.

Next, suppose $p = 3|n$. If $n = 3$ one has $h_{3,0} \cong K$, $h_{3,1} \cong \mathrm{sgn}$ and everything follows from Lemma 2.2. If $n \geq 6$ and $j \neq 3$ then, by Theorem 2.4, $H^1(\Sigma_n, (S^{(n-j,1^j)})^*) = 0$; the proof now is the same as for $p > 3$.

Finally, if $j = 3$ we have an exact sequence

$$0 \to h_{n,3} \to (S^{(n-3,1^3)})^* \to h_{n,2} \to 0.$$

This induces

$$H^0(\Sigma_n, h_{n,2}) \to H^1(\Sigma_n, h_{n,3}) \to H^1(\Sigma_n, (S^{(n-3,1^3)})^*) \to H^1(\Sigma_n, h_{n,2}),$$

the first and last members of which vanish. It remains to apply Theorem 2.4. □

Proposition 5.3. *Assume that $j \neq 3$ if $p = 3|n$. Then $H^2(\Sigma_n, h_{n,j}) = 0$ except the following cases:*

(1) $p|n$, $j = 2$;
(2) $p = 3$, $j = 6$;
(3) $p = 3, 3|n$, $j = 4$;
(4) $p = 3, 3 \nmid n$, $j = 3$;
(5) $n = 3$, $p = 3$, $j = 1$.

In the exceptional cases $\dim H^2(\Sigma_n, h_{n,j}) = 1$.

Proof. We have to consider several cases.

(a) $p \nmid n$. Then $h_{n,j} \cong S^{(n-j,1^j)} \cong (S^{(n-j,1^j)})^*$ and we can apply Theorem 4.1.

(b) $p|n, p > 3$. Here $H^1(\Sigma_n, (S^\lambda)^*) = H^2(\Sigma_n, (S^\lambda)^*) = 0$ for all λ by Theorems 2.4 and 4.1. So Proposition 5.1 implies

$$H^2(\Sigma_n, h_{n,j}) \cong H^1(\Sigma_n, h_{n,j-1}) = \begin{cases} 0, & j \neq 2, \\ K, & j = 2, \end{cases}$$

in view of Proposition 5.2, since $(S^{(n-j,1^j)}))^*/h_{n,j} \cong h_{n,j-1}$.

(c) $p = 3$, $n = 3$. Now $h_{3,0} \cong K$, $h_{3,1} \cong \mathrm{sgn}$ and we are done by Lemma 2.2.

(d) $p = 3|n$, $n \geq 6$, $j \in \{0, 1, \ldots, n-2\} \setminus \{3, 6\}$. According to Theorems 2.4 and 4.1, in this case one has $H^1(\Sigma_n, (S^\lambda)^*) = H^2(\Sigma_n (S^\lambda)^*) = 0$. By Proposition 5.1

$$H^2(\Sigma_n, h_{n,j}) \cong H^1(\Sigma_n, h_{n,j-1}) = \begin{cases} 0, & j \neq 2, 4, \\ K, & j = 2, 4. \end{cases}$$

(e) $p = 3|n, n \geq 9$, $j = 6$. From the sequence

$$0 \to h_{n,6} \to (S^{(n-6,1^6)})^* \to h_{n,5} \to 0,$$

we have

$$H^1(\Sigma_n, h_{n,5}) \to H^2(\Sigma_n, h_{n,6}) \to H^2(\Sigma_n, (S^{(n-6,1^6)})^*) \to H^2(\Sigma_n, h_{n,5}),$$

the first and last members of which vanish by Proposition 5.2 and part (d). Now apply Theorem 4.1. □

The proofs of the next three propositions require results from James [6] about decomposition matrices for 2-rowed diagrams and results from [9] about the 1-cohomology of the corresponding irreducibles. We give a brief reformulation for future reference.

Let $a, b \in \mathbb{Z}_+$, $a = \Sigma_{ni=0}^r a_i p^i$, $b = \Sigma_{j=0}^s b_i p^i$ their p-adic expansions. Assume that $a_r > 0$. Write $a \sqsupset_p b$ if $s < r$ and for all $k \leq s$ one has $b_k = a_k$ or 0.

(*) [6, 24.15] The Specht module $S^{(n-m,m)}$ is multiplicity-free and all its composition factors have the form $D^{(n-j,j)}$ for same $j \leq m$. Moreover, $D^{(n-j,j)}$ occurs as a composition factor of $S^{(n-m,m)}$ if and only if $n - 2j + 1 \sqsupset_p m - j$.

(**) [9, 4.7] Let $n + 1 = \Sigma_{i=0}^r a_i p^i$ be the p-adic expansion with $a_r > 0$. Then $H^1(\Sigma_n, D^{(n-j,j)}) = 0$ except the cases $j = a_i p^i$ where $i < r$ is such that $a_i > 0$. In the exceptional cases $\dim H^1(\Sigma_n, D^{(n-j,j)}) = 1$.

Proposition 5.4. *Suppose that* $m < p$. *Then* $H^2(\Sigma_n, D^{(n-m,m)}) = 0$, *except the cases*:

(1) $p = 3$, $n = 3$, $m = 1$ *and*

(2) $p = 3$, $n = 4$, $m = 2$,

where $\dim H^2(\Sigma_n, D^{(n-m,m)}) = 1$.

Proof. If $p = 3$, $D^{(2,1)} \cong \mathrm{sgn}_3$, $D^{(2^2)} \cong \mathrm{sgn}_4$, and in these cases one can apply Lemma 2.2. Otherwise, by Theorem 2.4 and Lemma 2.1, $H^1(\Sigma_n, (S^{(n-m,m)})^*) = H^2(\Sigma_n, (S^{(n-m,m)})^*) = 0$. Thus, Proposition 5.1 implies

$$H^2(\Sigma_n, D^{(n-m,m)}) \cong H^1(\Sigma_n, (J^{(n-m,m)})^*).$$

By (*), $D^{(n-j,j)}$ is a composition factor of $J^{(n-m,m)}$ if and only if $n - 2j + 1 \sqsupset_p m - j$. Since $m < p$ then $m - j < p$ and the last condition is equivalent to $n - 2j + 1 \equiv m - j \pmod p$, i.e. $j \equiv n + 1 - m \pmod p$. Define j_0 by

$$n + 1 - m = xp + j_0, \quad x \in \mathbb{Z}, \ 0 \leq j_0 < p.$$

If $j_0 < m$ then $J^{(n-m,m)} = D^{(n-j_0,j_0)}$. If $j_0 \geq m$ then $J^{(n-m,m)} = 0$. Hence assume $j_0 < m$. Define j_1 by

$$n + 1 = yp + j_1, \quad y \in \mathbb{Z}, \ 0 \leq j_1 < p.$$

In view of (**) one has $H^1(\Sigma_n, D^{(n-j,j)}) \neq {}^{\Gamma}$ for $0 \leq j < p$ implying $j = j_1$. Finally observe that $j_1 \neq j_0$ (since $0 < m < p$). □

Proposition 5.5. $\dim H^2(\Sigma_n, D^{(n-p,p)}) = 0$ *if* $n \equiv -1 \pmod p$ *and*

$$\dim H^2(\Sigma_n, D^{(n-p,p)}) = 1,$$

otherwise.

Proof. If $n \equiv -1 \pmod p$ the p-adic expansion of $n+1$ looks like $n + 1 = a_1 p + \cdots + a_r p^r$. According to (**) the inequality $\dim H^1(\Sigma_n, D^{(n-j,j)}) \neq 0$ implies $j = a_i p^i$ for $i > 0$ such that $a_i > 0$, in particular, $j \geq p$. Hence, in view of (*), $H^1(\Sigma_n, D^{(n-k,k)}) = 0$ for any factor $D^{(n-k,k)}$ of $J^{(n-p,p)}$, and by Proposition 5.1

$$H^2(\Sigma_n, D^{(n-p,p)}) \cong H^1(\Sigma_n, (J^{(n-p,p)})^*) = 0.$$

Let $n \not\equiv -1 \pmod p$. Then the p-adic expansion is $n + 1 = \Sigma\, a_i p^i$ with $a_0 > 0$. We want the composition factors of $J^{(n-p,p)}$. By (*), $D^{(n-j,j)}$ is a such a factor if and only if $j < p$ and $n + 1 - 2j \supset_p p - j$, i.e. $j \equiv n + 1 - p \pmod p$, which in turn is equivalent to $j = a_0$. Applying Proposition 5.1 yields

$$H^2(\Sigma_n, D^{(n-p,p)}) \cong H^1(\Sigma_n, D^{(n-a_0,a_0)}) \cong K$$

by (**). □

Remark. In general (meaning if $m > p$), the computation of $H^2(\Sigma_n, D^{(n-m,m)})$ appears to be a complicated task. Below we give a sufficient condition for its vanishing.

Proposition 5.6. *Let* $n + 1 = a_0 + a_1 p + \cdots + a_r p^r$, $a_r > 0$, *and* $m = m_0 + m_1 p + \cdots + m_r p^r$ *be p-adic expansions. We exclude the cases* $(p,n,m) = (3,3,1)$, $(3,4,2)$ (*for which see Proposition* 5.4). *Assume that* $H^2(\Sigma_n, D^{(n-m,m)}) \neq 0$. *Then one of the following two conditions holds*:

(i) *There exist integers* t, s *such that* $1 \leq t < s \leq r$; $a_t, a_s > 0$; $a_i = 0$ *for* $t < i < s$; $m_t = 0$; $m_s = a_s$ *or* 1; *for* $i \neq t, s$ *one has* $m_i = a_i$ *or* 0.

(ii) *There exist integers* s, t, u *with* $1 \leq t < u < s$; $a_t, a_s > 0$; $a_i = 0$ *for* $t < i < s$; $m_i = 0$ *for* $t \leq i < u$; $m_u = 1$; *for* $u < i < s$ *one has* $m_i = (p-1)$ *or* 0; $m_s = 0$ *or* $a_s - 1$; *for* $i \in \{0, 1, \ldots, t-1\} \cup \{s+1, \ldots, r\}$ *one has* $m_i = a_i$ *or* 0.

(iii) *There exist integers* t, s *such that* $1 \leq t < s \leq r$, $a_t, a_s > 0$, $a_i = 0$ *for* $t < i < s$, $m_i = a_i$ *or* 0 *for* $i \in [0, t-1] \cup [s+1, r]$; $m_t = a_t$; $m_i = 0$ *or* $p - 1$ *for* $i \in [t+1, s-1]$; $m_s = 0$ *or* $a_s - 1$.

Proof. By Proposition 5.1 $H^2(\Sigma_n, D^{(n-m,m)}) \neq 0$ implies $H^1(\Sigma_n, (J^{(n-m,m)})^*) \neq 0$, which in turn implies that $H^1(\Sigma_n, D^{(n-j,j)}) \neq 0$, for some composition factor $D^{(n-j,j)}$ of $(J^{(n-m,m)})^*$, hence of $J^{(n-m,m)}$.

From $(**)$, $j = a_t p^t$ for some $t < r$ with $a_{t r} > 0$. Let s be the smallest integer such that $s > t$ and $a_s > 0$ (which exists since $r > t$ and $a_r > 0$). The p-adic expansion of $n + 1 - 2j$ is now

$$\sum_{i=0}^{t-1} a_i p^i + (p - a_t)p^t + \sum_{i=t+1}^{s-1} (p-1)p^i + (a_s - 1)p^s + \sum_{i=s+1}^{r} a_i p^i.$$

Since $D^{(n-j,j)}$ is a factor of $J^{(n-m,m)}$, $(*)$ implies that $n + 1 - 2j \supset_p m - j$. If $m_t > 0$ then the tth component of the p-adic expansion of $m - j = m - a_t p^t$ cannot equal $(p - a_t)$. Hence $m_t = 0$ or $m_t = a_t$.

Assuming first that $m_t = 0$, let u be the smallest integer such that $u > t$ and $m_u > 0$ (which exists since $m > j$). The p-adic expansion of $m - j$ is now

$$\sum_{i=0}^{t-1} m_i p^i + (p - a_t)p^t + \sum_{i=t+1}^{u-1} (p-1)p^i + (m_u - 1)p^u + \sum_{i=u+1}^{r} m_i p^i.$$

The condition that $n + 1 - 2j \supset_p m - j$ implies that $u \le s$. If $u = s$ we obtain $m_s - 1 = 0$ or $m_s - 1 = a_s - 1$, and $m_i = a_i$ or 0 for $i \in \{0, 1, \ldots, t-1\} \cup \{s+1, \ldots, r\}$, giving (i).

If $u < s$ the $m_u = 1$, $m_i = (p-1)$ or 0 for $i \in \{u+1, \ldots, s-1\}$, $m_s = (a_s - 1)$ or 0 and again $m_i = a_i$ or 0 for $i \in \{0, 1, \ldots, t-1\} \cup \{s+1, \ldots, r\}$, giving (ii). The case where $m_t = a_t$ is considered similarly, yielding (iii). □

Proposition 5.7. *Let D be an irreducible Σ_n-module, with $D \ne D^{(n-2,1^2)}$ if $p \mid n$ and $H^1(\Sigma_{n-2}, D\downarrow_{\Sigma_{n-2}}) = H^1(\Sigma_{n-1}, D\downarrow_{\Sigma_{n-1}}) = H^1(\Sigma_n, D) = \mathrm{H}^2(\Sigma_{n-1}, D\downarrow_{\Sigma_{n-1}}) = 0$. Then $H^2(\Sigma_n, D) = 0$.*

Proof. From Theorem 3.2,

$$\begin{aligned}\dim H^2(\Sigma_n, D) &= \dim \mathrm{Hom}_{\Sigma_n}(S^{(n-2,1^2)}, D) + \dim \mathrm{Hom}_{\Sigma_n}(M^{(n-2,2)}, D)\\ &\quad - \dim \mathrm{Hom}_{\Sigma_n}(M^{(n)}, D) + \dim \mathrm{Hom}_{\Sigma_n}(M^{(n-1,1)}, D)\\ &\quad - \dim \mathrm{Hom}_{\Sigma_n}(M^{(n-2,1^2)}, D).\end{aligned}$$

The factors of $S^{(n-2,1^2)}$ and $D^{(n-2,1^2)}, D^{(n-1,1)}$ (the latter for $p \mid n$), and the factors of M^{μ} are some D^{ν} with $\nu \trianglerighteq \mu$ so we only have four cases to consider: $D = D^{(n)}$, $D^{(n-1,1)}, D^{(n-2,2)}, D^{(n-2,1^2)}$. The first is taken care of by Lemma 2.2. In the next two cases $H^2(\Sigma_n, D) = 0$ except for $p = 3$ and $\lambda = (2,1)$, (2^2), by Proposition 5.4. But $H^1(\Sigma_3, S^{(2,1)})^*) = H^1(\Sigma_3, \mathrm{sgn}) \ne 0$ by Lemma 2.2. (so the assumptions of Proposition 5.7 fail to hold). For the final case, $H^2(\Sigma_n, D^{(n-2,1^2)}) = 0$ if $p \nmid n$, by Theorem 4.1, since $D^{(n-2,1^2)} = S^{(n-2,1^2)}$. □

Remarks. (1) For $p \mid n$, $n > 3$, the irreducible module $D^{(n-2,1^2)}$ does provide the exception to Proposition 5.7 (cf. Proposition 5.3).

(2) It follows from Proposition 5.1 that if λ is p-regular and the trivial module $D^{(n)}$ does not occur as a composition factor of S^λ, then $H^1(\Sigma_n, D^\lambda) = 0$. Thus, using known decomposition matrices for small n, one can obtain results about the vanishing of $H^1(\Sigma_n, D^\lambda)$.

6. A remark on higher degree cohomologies

For semisimple algebraic groups dual Weyl modules are acyclic so the following question seems natural. For which i does the group $H^i(\Sigma_n,(S^\lambda)^*)$ vanish? Our final result gives an indirect answer to this question.

Proposition 6.1. *Fix p and let $i \in \mathbb{N}$. The following are equivalent.*
(i) $H^1(\Sigma_n,(S^\lambda)^*) = H^2(\Sigma_n,(S^\lambda)^*) = \cdots = H^i(\Sigma_n,(S^\lambda)^*) = 0$ *for any n and λ.*
(ii) $H^1(A_n,\mathbb{F}_p) = H^2(A_n,\mathbb{F}_p) = \cdots = H^i(A_n,\mathbb{F}_p) = 0$ *for any n.*

Proof. Note that $(1_{A_n})\uparrow^{\Sigma_n} \cong K \oplus \mathrm{sgn}$. So if $H^j(A_n,\mathbb{F}_p) \neq 0$ for some j with $1 \leq j \leq i$ then by Shapiro's Lemma

$$H^j(\Sigma_n, K \oplus \mathrm{sgn}) \cong H^j(\Sigma_n, K) \oplus H^j(\Sigma_n, \mathrm{sgn}) \neq 0.$$

But $K \cong (S^{(n)})^*$ and $\mathrm{sgn} \cong (S^{(1^n)})^*$.

In the other direction, if (ii) holds the same arguments show that $H^j(\Sigma_n, \mathrm{sgn}) = 0$ for any $n \geq 1$ and any j with $1 \leq j \leq i$. The proof of the equalities $H^j(\Sigma_n,(S^\lambda)^*) = 0$ is now completely analogous to that of Theorem 2.4 ($p > 3$) and Lemma 4.2, using the embedding $(S^\lambda)^* \hookrightarrow M^{\lambda'} \otimes \mathrm{sgn}$. $\square$

We end with a conjecture.

Conjecture 6.2. *For fixed i there exists some p_0 such that for $p > p_0$*

$$H^i(\Sigma_n,(S^\lambda)^*) = 0 \text{ for all } n \text{ and } \lambda.$$

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References

[1] K.S. Brown, Cohomology of Groups, Graduate Texts in Mathematics, Vol. 87 (Springer, New York, 1982).

[2] C.W. Curtis and I. Reiner, Representation Theory of Finite Groups and Associative Algebras (Wiley, New York, 1962).

[3] S. Donkin, On Schur algebras and related algebras II, J. Algebra 111 (1987) 354–364.
[4] D. Gorenstein, Finite Simple Groups (Plenum Press, New York, 1982).
[5] B. Huppert, Endliche Gruppen I (Springer, Berlin, 1967).
[6] G.D. James, The Representation Theory of the Symmetric Groups, Lecture Notes in Mathematics, Vol. 682 (Springer, Berlin, 1978).
[7] G.D. James, The representation theory of the symmetric groups, in: The Arcata Conf. on Repns. of Finite Groups. Proc. Sympos. Pure Math. 47 (1987) 111–126.
[8] J.C. Jantzen, Representations of Algebraic Groups (Academic Press, New York, 1987).
[9] A.S. Kleshchev and S. Martin, On the 1-cohomology of symmetric groups and semisimple algebraic groups, submitted.
[10] A.S. Kleshchev and A.A. Premet, On second degree cohomology of symmetric and alternating groups, Comm. Algebra 21(2) (1993) 583–600.
[11] M.H. Peel, Hook representations of the symmetric groups, Glasgow Math. J. 12 (1971) 136–149.